Discrete Mathematics

CMP-200

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Lecture 3

classification of compound propositions according to their possible truth values.

- ***Tautology***: a compound proposition that is always true
- ***Contradiction***: a compound proposition that is always false
- ***Contingency***: a compound proposition that is neither a tautology nor a contradiction

Tautologies, contradictions, contingencies

DEF: A compound proposition is called a ***tautology*** if no matter what truth values its atomic propositions have, its own truth value is **T**.

EG: $p \vee \neg p$ (Law of excluded middle)

The opposite to a tautology, is a compound proposition that's always false –a ***contradiction***.

EG: $p \wedge \neg p$

On the other hand, a compound proposition whose truth value isn't constant is called a ***contingency.***

EG: $p \rightarrow \neg p$

Tautologies and contradictions

The easiest way to see if a compound proposition is a tautology/contradiction is to use a truth table.

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

p	$\neg p$	$p \wedge \neg p$
F	T	F
T	F	F

Tautology example

Demonstrate that

$$[\neg p \wedge (p \vee q)] \rightarrow q$$

is a tautology in two ways:

1. Using a truth table – show that

$$[\neg p \wedge (p \vee q)] \rightarrow q$$

is always true

2. Using a proof (will get to this later).

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \to q$
T	T				
T	F				
F	T				
F	F				

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F			
T	F	F			
F	T	T			
F	F	T			

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T		
T	F	F	T		
F	T	T	T		
F	F	T	F		

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	
T	F	F	T	F	
F	T	T	T	T	
F	F	T	F	F	

Tautology by truth table

p	q	$\neg p$	$p \vee q$	$\neg p \wedge (p \vee q)$	$[\neg p \wedge (p \vee q)] \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Tautologies, contradictions and programming

Tautologies and contradictions in your code usually correspond to poor programming design.

EG:

```
while(x <= 3 || x > 3)
  x++;
```

```
if(x > y)
  if(x == y)
    return "never got here";
```

Logical Equivalences

DEF: Two compound propositions p, q are ***logically equivalent*** if their biconditional joining $p \leftrightarrow q$ is a tautology. Logical equivalence is denoted by $p \Leftrightarrow q$.

EG: The ***contrapositive*** of a logical implication is the reversal of the implication, while negating both components. I.e. the contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$. As we'll see next: $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$

Logical Equivalences

Remark: The symbol $\equiv$ is not a logical connective and $p \equiv q$ is not a compound proposition

but rather is the statement that $p \rightarrow q$ is a tautology. The symbol $\Leftrightarrow$ is sometimes used instead

of $\equiv$ to denote logical equivalence.

Logical Equivalence of Conditional and Contrapositive

The easiest way to check for logical equivalence is to see if the truth tables of both variants have *identical last columns*:

p	q	$p \rightarrow q$

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$

Q: why does this work given definition of $\Leftrightarrow$?

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p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$

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T	F	**F**
F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T			
T	F			
F	T			
F	F			

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F	F	**T**

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T	T	F		
T	F	T		
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p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T	F	F	
T	F	T	F	
F	T	F	T	
F	F	T	T	

Q: why does this work given definition of $\Leftrightarrow$?

Logical Equivalence of Conditional and Contrapositive

The easiest way to check for logical equivalence is to see if the truth tables of both variants have *identical last columns*:

p	q	$p \rightarrow q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

p	q	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T	F	F	**T**
T	F	T	F	**F**
F	T	F	T	**T**
F	F	T	T	**T**

Q: why does this work given definition of $\Leftrightarrow$?

Logical Equivalences

A: $p \Leftrightarrow q$ by definition means that $p \leftrightarrow q$ is a tautology. Furthermore, the biconditional is true exactly when the truth values of p and of q are identical. So if the last column of truth tables of p and of q is identical, the biconditional join of both is a tautology.

Logical Non-Equivalence of Conditional and Converse

The ***converse*** of a logical implication is the reversal of the implication. I.e. the converse of $p \rightarrow q$ is $q \rightarrow p$.

EG: The converse of "If Donald is a duck then Donald is a bird." is "If Donald is a bird then Donald is a duck."

As we'll see next: $p \rightarrow q$ and $q \rightarrow p$ are not logically equivalent.

Logical Non-Equivalence of Conditional and Converse

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \leftrightarrow (q \rightarrow p)$

Logical Non-Equivalence of Conditional and Converse

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \leftrightarrow (q \rightarrow p)$
T	T			
T	F			
F	T			
F	F			

Logical Non-Equivalence of Conditional and Converse

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \leftrightarrow (q \rightarrow p)$
T	T	T		
T	F	F		
F	T	T		
F	F	T		

Logical Non-Equivalence of Conditional and Converse

p	q	$p \to q$	$q \to p$	$(p \to q) \leftrightarrow (q \to p)$
T	T	T	T	
T	F	F	T	
F	T	T	F	
F	F	T	T	

Logical Non-Equivalence of Conditional and Converse

p	q	$p \to q$	$q \to p$	$(p \to q) \leftrightarrow (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Logical Equivalence

Equivalence	***Name***
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law

Logical Equivalence

$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws

Logical Equivalence

$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv \mathbf{T}$ $p \land \neg p \equiv \mathbf{F}$	Negation laws

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Logical Equivalences Involving Conditional Statements.

$$p \to q \equiv \neg p \lor q$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \lor q \equiv \neg p \to q$$

$$p \land q \equiv \neg(p \to \neg q)$$

$$\neg(p \to q) \equiv p \land \neg q$$

$$(p \to q) \land (p \to r) \equiv p \to (q \land r)$$

$$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$$

$$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$$

$$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$$

Logical Equivalences Involving Biconditionals.

$$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Example

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

$\neg(p \vee (\neg p \wedge q))$	$\equiv \neg p \wedge \neg(\neg p \wedge q)$	by the second De Morgan law
	$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$	by the first De Morgan law
	$\equiv \neg p \wedge (p \vee \neg q)$	by the double negation law
	$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
	$\equiv \mathbf{F} \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv \mathbf{F}$
	$\equiv (\neg p \wedge \neg q) \vee \mathbf{F}$	by the commutative law for disjunction
	$\equiv \neg p \wedge \neg q$	by the identity law for **F**

Predicates

Statements involving variables, such as

and

"x > 3 , " "x = y + 3," "x + y = z,"

"computer x is under attack by an intruder,"

"computer x is functioning properly,“

These statements are neither true nor false when the values of the variables are not specified.

Predicates

Propositions can be produced from such statements.

The statement "x is greater than 3" has two parts.

- The first part, the variable x, is the subject of the statement.
- The second part-the predicate, "is greater than 3"

Predicates

We can denote the statement "x is greater than 3" by P (x) where P denotes the predicate "is greater than 3" and x is the variable.

The statement P(x) is also said to be the value of the propositional function P at x.

Once a value has been assigned to the variable x , the statement P (x) becomes a proposition and has a truth value.

Predicates

Example

Let P(x) denote the statement "x > 3." What are the truth values of P (4) and P (2)?

P (4), which is the statement "4 > 3," is true.

P (2), which is the statement "2 > 3," is false.

Predicates

Example

Let Q(x , y) denote the statement "x = y + 3 ." What are the truth values of the propositions Q(1 , 2) and Q(3, 0)?

Q (1 , 2) is the statement " 1 = 2 + 3," which is false.

Q (3, 0) is the proposition "3 = 0 + 3,“ which is true.

Predicates

Example

Let A(c, n) denote the statement "Computer c is connected to network n ," where c is a variable representing a computer and n is a variable representing a network. Suppose that the computer MATH I is connected to network CAMPUS2, but not to network CAMPUS l . What are the

values of

A(MATH I , CAMPUS 1)

A(MATH I , CAMPUS2)?

Predicates

In general, a statement involving the n variables can be denoted by

$P(x_1, x_2, \ldots, x_n)$

P is also called a n -place predicate or a n-ary predicate

Quantifiers

- Quantification

Expresses the extent to which a predicate is true over a range of elements.

In English, the words all, some, many, none, and few are used in quantifications.

Quantification

- Only two types:
  - **Universal quantification:** a predicate is true for every element
  - **Existential quantification:** there is one or more element for which a predicate is true

The Universal Quantifier

- Domain: domain of discourse (universe of discourse)
- Definition 1: The *universal quantification* of *P(x)* is the statement "*P(x)* for all values of x in the domain", denoted by $\forall x\, P(x)$
  - "for all *x P(x)*" or "for every *x P(x)*"
    - Counterexample: an element for which *P(x)* is false
  - When all elements in the domain can be listed, $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$

Example of Universal Quantifier

Let P (x) be the statement

"x + 1 > x "

What is the truth value o f the quantification $\forall x\, P(x)$ where the domain consists of all real numbers?

Because P (x) is true for all real numbers x, the quantification $\forall x\, P(x)$ is true.

Example of Universal Quantifier

- What is the truth value of $\forall x\, P(x)$, where P (x) is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4)$$

Example of Universal Quantifier

- What is the truth value of $\forall x\ (x^2 \geq x)$ if the domain consists of all real numbers?
- What is the truth value of this statement if the domain consists of all integers?

What if $x = \frac{1}{2}$

Existential Quantification

- Definition: The *existential quantification* of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$", denoted by

  $\exists x\ P(x)$

  - "there is an x such that $P(x)$" or "for some x $P(x)$"
  - When all elements in the domain can be listed, $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

Example of Existential Quantifier

- Let P (x) denote the statement "x > 3 ." What is the truth value of the quantification $\exists x\, P(x)$

where the domain consists of all real numbers?

- Because "x > 3" is sometimes true-for instance, when x = 4, the existential quantification of P (x), which is $\exists x\, P(x)$, is true.

Negating Quantified Expressions

- We will often want to consider the negation of a quantified expression

- For instance, consider the negation of the statement

Every student in your class has taken a course in programming.

Negating Quantified Expressions

- This statement is a universal quantification, namely,

$$\forall x\ P(x)$$

- The negation of this statement is "It is not the case that Every student in your class has taken a course in programming"

Negating Quantified Expressions

- This is equivalent to "There is a student in your class who has not taken a course in calculus." And this is simply the existential quantification of the negation of the original propositional function, namely,

$$\exists x \neg P(x)$$

- This example illustrates the following logical equivalence

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

Negating Quantified Expressions

What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

Solution: The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$, which is equivalent to $\exists x\neg(x^2 > x)$. This can be rewritten as $\exists x(x^2 \leq x)$. The negation of $\exists x(x^2 = 2)$ is the statement $\neg\exists x(x^2 = 2)$, which is equivalent to $\forall x\neg(x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$. The truth values of these statements depend on the domain. ◀