Discrete Mathematics

CMP-101

Lecture 10

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Outline

- Sequences
- Summations
- Countable Sets
- Uncountable Sets

Sequences

- A sequence represents an ordered list of elements.

  e.g., $\{a_n\} = 1, 1/2, 1/3, \quad 1/4, \ldots$

- Formally: A *sequence* or *series* $\{a_n\}$ is identified with a *generating function* $f{:}S \rightarrow A$ for some subset $S \subseteq \mathbf{N}$ (often $S = \mathbf{N}$) and for some set A.

  e.g., $a_n = f(n) = 1/n$.

- The symbol a_n denotes $f(n)$, also called *term* n of the sequence.

Example with Repetitions

- Consider the sequence $b_n = (-1)^n$.
- $\{b_n\} = 1, -1, 1, -1, \ldots$
- $\{b_n\}$ denotes an infinite sequence of 1's and –1's, *not* the 2-element set $\{1, -1\}$.

Geometric Progression

- A geometric progression is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term a* and the *common ratio r* are real numbers

A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

Example

- The sequences $\{b_n\}$ with $b_n = (-1)^n$, is a geometric progressions with initial term and common ratio equal to 1 and - 1 if we start at n = 0
- The list of terms b_0, b_1 , b_2 , b_3 , b_4 , • • • begins with 1 , - 1 , 1 , - 1 , 1 , . . .

Example

- The sequences $\{c_n\}$ with $c_n=2\cdot5^n$, is a geometric progressions with initial term and common ratio equal to 2 and 5, if we start at n = 0
- the list of terms c_0 , c_1 , c_2 , c_3 , c_4 , • • • begins with 2, 10 , 50, 250, 1250, . . . ;

Example

- The sequences $\{d_n\}$ with $d_n = 6\cdot(l/3)^n$ is a geometric progressions with initial term and common ratio equal to 6 and 1/3, respectively, if we start at n = 0.
- The list of terms d_0 , d_l , d_2 , d_3 , d_4 ,…. begins with
- 6, 2 , 2/3 , 2/9 , 2/27 , ……..

Arithmetic Progression

- A *arithmetic progression* is a sequence of the form
- *a, a+d, a+2d, …, a+nd, …*

  where the *initial term a* and the *common difference d* are real numbers
- An arithmetic progression is a discrete analogue of the linear function f(x) = dx + a

Example

- The sequences $\{S_n\}$ with $S_n = -1 + 4n$ is an arithmetic progressions with initial terms and common differences equal to -1 and 4 if we start at $n = 0$.
- The list of terms $S_0, S_1, S_2, S_3, \ldots$ begins with $-1, 3, 7, 11, \ldots,$

Example

- The sequences $\{t_n\}$ with $t_n = 7 - 3n$ is both arithmetic progressions with initial terms and common differences equal to 7 and -3 if we start at $n = 0$.
- The list of terms $t_0, t_1, t_2, t_3, \ldots$ begins with 7, 4, 1, -2, ...

Strings

- Sequences of the form a_1 , a_2 , . . . , a_n are often used in computer science. These finite sequences are also called strings. This string is also denoted by $a_1a_2...a_n$. The length of the string S is the number of terms in this string. The empty string, denoted by λ , is the string that has no terms. The empty string has length zero.
- Example: The string abcd is a string of length four

Recognizing Sequences

- Sometimes, you're given the first few terms of a sequence, and you are asked to find the sequence's generating function, or a procedure to enumerate the sequence.

Recognizing Sequences

- Examples: What's the next number?
  - 1,2,3,4,… **5** (the 5th smallest number >0)
  - 1,3,5,7,9,… **11** (the 6th smallest odd number >0)
  - 2,3,5,7,11,... **13** (the 6th smallest prime number)

The Trouble with Recognition

- The problem of finding "the" generating function given just an initial subsequence is *not well defined*.
- This is because there are *infinitely* many computable functions that will generate *any* given initial subsequence.

The Trouble with Recognition (cont'd)

- We implicitly are supposed to find the *simplest* such function but, how should we define the *simplicity* of a function?
  - We might define simplicity as the reciprocal of complexity, but…
  - There are *many* reasonable, competing definitions of complexity, and this is an active research area.
- So, these questions really have *no* objective right answer!

Exercise

- Ex. 1, 3, 5, 7, 9, …
- Ex. 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, …
- Ex. 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047, …

Important:

- Another useful technique for finding a rule for generating the terms of a sequence is to compare the terms of a sequence of interest with the terms of a well-known integer sequence, such as terms of arithmetic progression, terms of a geometric progression, perfect squares, perfect cubes, and so on. The first 10 terms of some sequences you may want to keep in mind are displayed in Table

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...

Summations

- Given a series $\{a_n\}$, the *summation of* $\{a_n\}$ *from j to k* is written and defined as follows:

$$\sum_{i=j}^{k} a_i :\equiv a_j + a_{j+1} + \ldots + a_k$$

- Here, *i* is called the *index of summation*.
- *j:* lower limit
- *k:* upper limit

- Express the sum of the first 100 terms of the sequence {an }, where $a_n = 1 / n$ for n = 1 , 2, 3 , ...

$$\sum_{j=1}^{100} \frac{1}{j}$$

Example

- What is the value of

$$\sum_{j=1}^{5} j^2$$

Example

- What is the value of

$$\sum_{k=4}^{8}(-1)k$$

Shifting Index of Summation:

- Sometimes it is useful to shift the index of summation in a sum. This is often done when two sums need to be added but their indices of summation do not match. When shifting an index of summation, it is important to make the appropriate changes in the corresponding summand.

Example

Suppose we have the sum

$$\sum_{j=1}^{5} j^2$$

But want the index of summation to run between 0 and 4 rather than from 1 to 5 .
To do this, we let k= j - 1 . Then the new summation index runs from 0 to 4,
and the term j^2 becomes $(k + 1)^2$. Hence

$$\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4} (k+1)^2$$

Theorem 1 (*Geometric Series*)

If *a* and *r* are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^{j} = \begin{cases} \dfrac{ar^{n+1} - a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

Double Summation:

- Double summations arise in many contexts (as in the analysis of nested loops in computer programs). An example of a double summation is

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij.$$

- To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation:

Double Summation:

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i)$$

$$= \sum_{i=1}^{4} 6i$$

$$= 6 + 12 + 18 + 24 = 60.$$

Summation over Sets

- We can also use summation notation to add all values of a function, or terms of an indexed set, where the index of summation runs over all values in a set. That is, we write

$$\sum_{s \in S} f(s)$$

to represent the sum of the values f(s), for all members s of S.

Example

What is the value of $\sum_{s \in \{0,2,4\}} s$?

Solution: Because $\sum_{s \in \{0,2,4\}} s$ represents the sum of the values of s for all the members of the set $\{0, 2, 4\}$, it follows that

$$\sum_{s \in \{0,2,4\}} s = 0 + 2 + 4 = 6.$$

◀

Summation Formulae

- Certain sums arise repeatedly throughout discrete mathematics. Having a collection of formulae for such sums can be useful; Table 2 provides a small table of formulae for commonly occurring sums.

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty}, kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Cardinality

- We defined the cardinality of a finite set to be the number of elements in the set.
- The cardinality of a finite set tells us when two finite sets are the same size, or when one is bigger than the other.
- There is a one-to-one correspondence between any two finite sets with the same number of elements.
- This observation lets us extend the concept of cardinality to all sets, both finite and infinite,

Same Cardinality:

- The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B .
- We will now split infinite sets into two groups, those with the same cardinality as the set of natural numbers and those with different cardinality.

Countable Sets

- A set that is either finite or has the same cardinality as the set of positive integers is called countable.
- When an infinite set S is countable, $|S|=\aleph_0$ ("aleph null")

Uncountable Sets

- A set that is not countable is called *uncountable*