Discrete Mathematics

CMP-200

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Nested Quantifiers

Two quantifiers are nested if one is within the scope of the other, such as

$$\forall x\ \exists y\ (x+y=0)$$

Note that everything within the scope of a quantifier can be thought of as a propositional function

Explanation

To understand these statements involving many quantifiers, we need to unravel what the quantifiers and predicates that appear mean.

Some examples to illustrate

Example 1

Assume that the domain for the variables x and y consists of all real numbers

$$\forall x \ \forall y(x+y=y+x)$$

This is the commutative law for addition of real numbers. Likewise, the statement

Example 2

$$\forall x\ \exists y\ (x+y=0)$$

For every real number x there is a real number y such that x + y = 0. This states that every real number has an additive inverse.

Example 2

$$\forall x \, \forall y \, \forall z \, (x + (y + z) = (x + y) + z)$$

The associative law for addition of real numbers.

Translate into English

$$\forall x\ \forall y\ ((x>0) \wedge (y<0) \rightarrow (xy<0))$$

where the domain for both variables consists of all real numbers

- This statement says that for every real number x and for every real number y, if $x>0$ and $y<0$, then $xy<0$. That is, for real numbers x and y, if x is positive and y is negative, then xy is negative.
- The product of a positive real number and a negative real number is always a negative real number

Thinking of quantification as loops

Think in terms of nested loops

- Of course, if there are infinitely many elements in the domain of some variable, we cannot actually loop through all values.

- For example, to see whether $\forall x \forall y P(x, y)$ is true, we loop through the values for x , and for each x we loop through the values for y. If we find that P (x , y) is true for all values for x and y, we have determined that $\forall x \forall y P(x, y)$ is true

Thinking of quantification as loops

Translate

- $\forall x \, \exists y \, P(x, y)$
- $\exists x \, \forall y \, P(x, y)$
- $\exists x \, \exists y \, P(x, y)$

Thinking of quantification as loops

$$\forall x \, \exists y \, P(x, y)$$

We loop through the values for x , and for each x we loop through the values for y. If we find that P (x , y) is true for all values for x and y, we have determined that $\forall x \exists y P(x, y)$ is true. if for some x we never hit such a y, then $\forall x \exists y P(x, y)$ is false.

Thinking of quantification as loops

$\exists x \forall y \, P(x, y)$

We loop through the values for x until we find an x for which P (x , y) is always true when we loop through all values for y. Once we find such an x , we know that $\exists x \forall y \, P(x,y)$ is true. If we never hit such an x , then we know that $\exists x \forall y \, P(x, y)$ is false.

Thinking of quantification as loops

$$\exists x \, \exists y \, P(x, y)$$

we loop through the values for x, where for each x we loop through the values for y until we hit an x for which we hit a y for which P (x , y) is true. The statement $\exists x \, \exists y$ P (x , y) is false only if we never hit an x for which we hit a y such that P (x , y) is true.

Practical Examples

Let Q (x , y) be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

$$\exists x\, \exists y\, Q(x, y)$$

There is some student in the class who has sent an email message to some student in the class

Practical Examples

Let Q (x , y) be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

$$\exists x \ \forall y \ Q(x, y)$$

There is some student in the class who has sent an email message to every student in the class

Practical Examples

Let Q (x , y) be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

$$\exists y \; \forall x \; Q(x, y)$$

There is a student in your class who has been sent an email message by every student of your class.

Practical Examples

Let Q (x , y) be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

$$\forall y \, \exists x \, Q(x, y)$$

Every student in your class has been sent an email message from at least one student in your class.

Practical Examples

Let Q (x , y) be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

$$\forall x \ \forall y \ Q(x, y)$$

Every student in your class have sent an email message to every student in your class.

Practical Examples

Let C (x , y) mean that student x is enrolled in class y, where the domain for x consists of all students in your school and the domain for y consists of all classes being given at your school. Express each of these statements by a simple English sentence.

C (Randy Goldberg, CS 252)

Randy Goldberg is enrolled in CS 252.

Practical Examples

Let C (x , y) mean that student x is enrolled in class y, where the domain for x consists of all students in your school and the domain for y consists of all classes being given at your school. Express each of these statements by a simple English sentence.

$$\exists x\, C(x, \mathit{Math}\ 695)$$

Someone is enrolled in Math 695

Practical Examples

Let C (x , y) mean that student x is enrolled in class y, where the domain for x consists of all students in your school and the domain for y consists of all classes being given at your school. Express each of these statements by a simple English sentence.

$$\exists\, yC\,(Carol\ Sitea, y)$$

Carol Sitea is enrolled in some course.

Practical Examples

Let C (x , y) mean that student x is enrolled in class y, where the domain for x consists of all students in your school and the domain for y consists of all classes being given at your school. Express each of these statements by a simple English sentence.

$$\exists\, 3x\, (C\,(x\,,\, Math\ 222) \wedge C\,(x\,,\, CS\ 252))$$

Some student is enrolled simultaneously in Math 222 and CS 252

Practical Examples

Let Q (x , y) be the statement "student x has been a contestant on quiz show y ." Express each of these sentences in terms of Q (x , y), quantifiers, and logical connectives, where the domain for x consists of all students at your school and for y consists of all quiz shows on television.

There is a student at your school who has been a contestant on a television quiz show.

$$\exists x \exists y Q(x, y)$$

Practical Examples

Let Q (x , y) be the statement "student x has been a contestant on quiz show y ." Express each of these sentences in terms of Q (x , y), quantifiers, and logical connectives, where the domain for x consists of all students at your school and for y consists of all quiz shows on television.

No student at your school has ever been a contestant on a television quiz show.

$$\neg\exists x\exists y Q(x, y) \text{ or } \forall x\forall y\neg Q(x, y).$$

Practical Examples

Let Q (x , y) be the statement "student x has been a contestant on quiz show y ." Express each of these sentences in terms of Q (x , y), quantifiers, and logical connectives, where the domain for x consists of all students at your school and for y consists of all quiz shows on television.

There is a student at your school who has been a contestant on Jeopardy and on Wheel of Fortune.

$$\exists x(Q(x, \text{Jeopardy}) \land Q(x, \text{Wheel of Fortune}))$$

Practical Examples

Let Q (x , y) be the statement "student x has been a contestant on quiz show y ." Express each of these sentences in terms of Q (x , y), quantifiers, and logical connectives, where the domain for x consists of all students at your school and for y consists of all quiz shows on television.

Every television quiz show has had a student from your school as a contestant.

$$\forall y \exists x Q(x, y)$$

Practical Examples

Let Q (x , y) be the statement "student x has been a contestant on quiz show y ." Express each of these sentences in terms of Q (x , y), quantifiers, and logical connectives, where the domain for x consists of all students at your school and for y consists of all quiz shows on television.

At least two students from your school have been contestants on Jeopardy.

$$\exists x_1 \exists x_2 (Q(x_1, \text{Jeopardy}) \land Q(x_2, \text{Jeopardy}) \land x_1 \neq x_2)$$

Order of Quantifiers

- The order of quantifiers is important unless all quantifiers are universal quantifiers or all are existential quantifiers
  - $\forall x \forall y\, P(x, y)$ vs. $\forall y \forall x\, P(x, y)$
    - $P(x,y)$: "$x+y=y+x$"
  - $\forall x \exists y\, Q(x, y)$ vs. $\exists y \forall x\, Q(x, y)$
    - $Q(x,y)$: "$x+y=0$"

Order of Quantifiers

Statement	*When True?*	*When False?*
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x,y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which $P(x,y)$ is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x,y for which $P(x,y)$ is true.	$P(x,y)$ is false for every pair x,y.

Order of Quantifiers

Let $Q(x, y, z)$ be the statement "$x + y = z$." What are the truth values of the statements $\forall x \forall y \exists z Q(x, y, z)$ and $\exists z \forall x \forall y Q(x, y, z)$, where the domain of all variables consists of all real numbers?

Suppose that x and y are assigned values. Then, there exists a real number z such that x + y = z. Consequently, the quantification

$$\forall x \forall y \exists z Q(x, y, z),$$

which is the statement "For all real numbers x and for all real numbers y there is a real number z such that x + y = z," is true.

Order of Quantifiers

The order of the quantification here is important, because the quantification

$$\exists z \forall x \forall y \, Q(x, y, z),$$

which is the statement "There is a real number z such that for all real numbers x and for all real numbers y it is true that x + y = z is false, because there is no value of z that satisfies the equation x + y = z for all values of x and y.

Translating Mathematical Statements into Statements Involving Nested Quantifiers

Translate the statement "The sum of two positive integers is always positive" into a logical expression.

First rewrite it so that the implied quantifiers and a domain are shown: "For every two integers, if these integers are both positive, then the sum of these integers is positive.“

Next, we introduce the variables x and y to obtain "For all positive integers x and y, x + y is positive."

$$\forall x \forall y ((x > 0) \land (y > 0) \rightarrow (x + y > 0))$$

Translating Mathematical Statements into Statements Involving Nested Quantifiers

Translate the statement "Every real number except zero has a multiplicative inverse."

We first rewrite this as "For every real number x except zero, x has a multiplicative inverse." We can rewrite this as "For every real number x , if x ≠ 0, then there exists a real number y such that x y = 1"

$$\forall x((x \neq 0) \rightarrow \exists y(xy = 1)).$$

Translating from Nested Quantifiers into English

$$\forall x(C(x) \vee \exists y(C(y) \wedge F(x, y)))$$

where C (x) is "x has a computer," F (x , y) is "x and y are friends," and the domain for both x and y consists of all students in your school.

For every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends. In other words, every student in your school has a computer or has a friend who has a computer

Negating Nested Quantifiers

Statements involving nested quantifiers can be negated by successively applying the rules for negating statements involving a single quantifier.

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Negating Nested Quantifiers

Express the negation of the statement $\forall x \exists y(xy = 1)$ so that no negation precedes a quantifier.

Solution: By successively applying De Morgan's laws for quantifiers in Table 2 of Section 1.3, we can move the negation in $\neg\forall x \exists y(xy = 1)$ inside all the quantifiers. We find that $\neg\forall x \exists y(xy = 1)$ is equivalent to $\exists x \neg\exists y(xy = 1)$, which is equivalent to $\exists x \forall y \neg(xy = 1)$. Because $\neg(xy = 1)$ can be expressed more simply as $xy \neq 1$, we conclude that our negated statement can be expressed as $\exists x \forall y(xy \neq 1)$. ◀

Negating Nested Quantifiers

Use quantifiers to express the statement` that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: This statement is the negation of the statement "There is a woman who has taken a flight on every airline in the world" our statement can be expressed as $\neg\exists w \forall a \exists f(P(w, f) \wedge Q(f, a))$, where $P(w, f)$ is "w has taken f" and $Q(f, a)$ is "f is a flight on a." By successively applying De Morgan's laws for quantifiers in Table 2 of Section 1.3 to move the negation inside successive quantifiers and by applying De Morgan's law for negating a conjunction in the last step, we find that our statement is equivalent to each of this sequence of statements:

$$\begin{aligned}\forall w \neg\forall a \exists f(P(w, f) \wedge Q(f, a)) &\equiv \forall w \exists a \neg\exists f(P(w, f) \wedge Q(f, a)) \\ &\equiv \forall w \exists a \forall f \neg(P(w, f) \wedge Q(f, a)) \\ &\equiv \forall w \exists a \forall f(\neg P(w, f) \vee \neg Q(f, a)).\end{aligned}$$

This last statement states "For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline." ◀