Discrete Mathematics

CMP-200

Lecture 5
Rules of Inference, Building Argument, Fallacies, Introduction to Proofs

Abdul Hameed

http://informationtechnology.pk/pucit

abdul.hameed@pucit.edu.pk

Inference Rules and Formal Proofs

Proofs in mathematics are **valid arguments** that establish the truth of mathematical statements.

An argument is a sequence of statements that end with a conclusion.

The argument is valid if the conclusion (Final statement) follows from the truth of the preceding statements (premises).

Rules of inference are templates for building valid arguments.

Valid Arguments using Propositional Logic

Consider the following argument (sequence of propositions):

- If the prof offers chocolate for an answer, you answer the prof's question.
- The prof offers chocolate for an answer.
- Therefore, you answer the prof's question.

Let p be "the prof offers chocolate for an answer"
and q be "you answer the prof's question".

The *form* of the above argument is:

$$\begin{array}{rl} & p \rightarrow q \\ & p \\ \hline \therefore & q \end{array}$$

The argument is valid since $((p \rightarrow q) \wedge p) \rightarrow q$ is a tautology.

Arguments, argument forms and their validity

Definition

An *argument* in propositional logic is sequence of propositions. All but the final proposition are called *premises* and the final proposition is called the *conclusion*. An argument is *valid* if the truth of all its premises implies that the conclusion is true.

An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* if no matter which propositions are substituted for the propositional variables in its premises, if the premises are all true, then the conclusion is true.

In other words, an argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion q is valid if and only if

$$(p_1 \land p_2 \land \cdots \land p_n) \to q$$

is a tautology.

Valid Arguments in Propositional Logic

- "If you have a current password, then you can log onto the network"
- "You have a current password"
- Therefore, "You can log onto the network"
- $$\begin{array}{l} p \rightarrow q \\ p \\ \therefore q \end{array}$$

- *Argument*: a sequence of propositions
  - *Premises*
  - *Conclusion:* the final proposition
  - *Argument form*: a sequence of compound propositions involving propositional variables

inference rule	tautology	name
p $p \rightarrow q$ $\therefore\ q$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens (mode that affirms)
$\neg q$ $p \rightarrow q$ $\therefore\ \neg p$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus tollens (mode that denies)
$p \rightarrow q$ $q \rightarrow r$ $\therefore\ p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore\ q$	$((p \vee q) \wedge (\neg p)) \rightarrow q$	disjunctive syllogism

Rules of Inference for Propositional Logic II

p $\therefore\ p \lor q$	$p \rightarrow (p \lor q)$	addition
$p \land q$ $\therefore\ p$	$(p \land q) \rightarrow p$	simplification
p q $\therefore\ p \land q$	$((p) \land (q)) \rightarrow (p \land q)$	conjunction
$p \lor q$ $\neg p \lor r$ $\therefore\ q \lor r$	$((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$	resolution

Example of Building Arguments

"It it not sunny this afternoon and it is colder than yesterday"

"We will go swimming only if it is sunny"

"If we do not go swimming, then we will take a canoe trip"

"If we take a canoe trip, then we will be home by sunset"

Conclusion: "We will be home by sunset"

Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If Socrates is human, then Socrates is mortal.

Socrates is human.

∴ Socrates is mortal.

Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

If George does not have eight legs, then he is not an insect.

George is an insect.

∴ George has eight legs.

What rule of inference is used in each of these arguments?

Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

$\begin{array}{rl} & p \\ \hline \therefore & p \vee q \end{array}$	$p \rightarrow (p \vee q)$	addition

What rule of inference is used in each of these arguments?

Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.

$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$	$(p \wedge q) \rightarrow p$	simplification

What rule of inference is used in each of these arguments?

If it is rainy, then the pool will be closed. It is rainy.

Therefore, the pool is closed.

p $p \rightarrow q$ $\therefore\ q$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens (mode that affirms)

What rule of inference is used in each of these arguments?

- If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.

$\begin{array}{rl} & \neg q \\ & p \rightarrow q \\ \hline \therefore & \neg p \end{array}$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus tollens (mode that denies)

Fallacies

$((p \to q) \wedge q) \to p$ is not a tautology

Fallacy of affirming the conclusion

"If you do every problem in this book, then you will learn discrete mathematics. You did every problem in this book Therefore,."

$((p \to q) \wedge \neg p) \to \neg q$ is not a tautology

Fallacy of denying the hypothesis

Introduction to the Proofs

A proof is a valid argument that establishes the truth of a mathematical statement, using the hypotheses of the theorem, if any, axioms assumed to be true, and previously proven theorems.

Using these ingredients and rules of inference, the proof establishes the truth of the statement being proved.

Some terminology

Theorem: a statement that can be shown to be true (sometimes referred to as facts or results). Less important theorems are often called propositions.

A **lemma** is a less important theorem, used as an auxiliary result to prove a more important theorem.

A **corollary** is a theorem proven as an easy consequence of a theorem.

A **conjecture** is a statement that is being proposed as a true statement. If later proven, it becomes a theorem, but it may be false.

Some terminology

Axiom (or postulates) are statements that we assume to be true (algebraic axioms specify rules for arithmetic like commutative laws).

A proof is a valid argument that establishes the truth of a theorem. The statements used in a proof include axioms, hypotheses (or premises), and previously proven theorems. Rules of inference, together with definition of terms, are used to draw conclusions from other assertions, tying together the steps of a proof.

Understanding how theorems are stated

Many theorems assert that a property holds for all elements in a domain. However, the universal quantifier is often not explicitly stated.

The statement:

"If $x > y$, where x and y are positive real numbers, then $x^2 > y^2$."

really means

"For all positive real numbers x and y, if $x > y$ then $x^2 > y^2$."

That is, in formal logic under the domain of positive real numbers this is the same as $\forall x \forall y((x > y) \rightarrow (x^2 > y^2))$.