Discrete Mathematics

CMP-200

Lecture 6

Abdul Hameed

http://informationtechnology.pk/pucit

abdul.hameed@pucit.edu.pk

Direct Proofs

We will learn:

- Methods of Proving Theorems.
  - Direct Proofs
  - Proof by Contraposition
  - Proofs by Contradiction

Methods of Proving Theorems

- Proving theorems can be difficult.
- Ammunition
- To prove a theorem of the form

$$\forall x\,(P(x) \rightarrow Q(x)),$$

our goal is to show that $P(x) \rightarrow Q(x)$ is true, where c is an arbitrary element of the domain, and then apply universal generalization

Direct Proof

- A direct proof of a conditional statement $p \rightarrow q$ *is* constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true.

- A direct proof shows that a conditional statement $p \rightarrow q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs.

Direct Proof

- Definition 1: The integer *n* is *even* if there exists an integer *k* such that $n=2k$, and *n* is *odd* if there exists an integer *k* such that $n=2k+1$.

Exercise

Prove that "if n is an odd integer, then n^2 is odd."

Exercise

Solution: Note that this theorem states $\forall n P((n) \rightarrow Q(n))$, where $P(n)$ is "n is an odd integer" and $Q(n)$ is "n^2 is odd." As we have said, we will follow the usual convention in mathematical proofs by showing that $P(n)$ implies $Q(n)$, and not explicitly using universal instantiation. To begin a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that n is odd. By the definition of an odd integer, it follows that $n = 2k + 1$, where k is some integer. We want to show that n^2 is also odd. We can square both sides of the equation $n = 2k + 1$ to obtain a new equation that expresses n^2. When we do this, we find that $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. By the definition of an odd integer, we can conclude that n^2 is an odd integer (it is one more than twice an integer). Consequently, we have proved that if n is an odd integer, then n^2 is an odd integer. ◀

Exercise

Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square

(An integer a is a perfect square if there is an integer b such that $a = b^2$)

Exercise

Solution: To produce a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that m and n are both perfect squares. By the definition of a perfect square, it follows that there are integers s and t such that $m = s^2$ and $n = t^2$. The goal of the proof is to show that mn must also be a perfect square when m and n are; looking ahead we see how we can show this by multiplying the two equations $m = s^2$ and $n = t^2$ together. This shows that $mn = s^2t^2$, which implies that $mn = (st)^2$ (using commutativity and associativity of multiplication). By the definition of perfect square, it follows that mn is also a perfect square, because it is the square of st, which is an integer. We have proved that if m and n are both perfect squares, then mn is also a perfect square. ◀

Exercise

Use a direct proof to show that the sum of two even integers is even.

We must show that whenever we have two even integers, their sum is even. Suppose that a and b are two even integers. Then there exist integers s and t such that $a = 2s$ and $b = 2t$. Adding, we obtain $a + b = 2s+2t = 2(s + t)$. Since this represents $a + b$ as 2 times the integer $s + t$, we conclude that $a + b$ is even, as desired.

Exercise

Show that the additive inverse, or negative, of an even number is an even number using a direct proof.

- We must show that whenever we have an even integer, its negative is even. Suppose that a is an even integer. Then there exists an integer s such that $a = 2s$. Its additive inverse is $-2s$, which by rules of arithmetic and algebra equals $2(-s)$. Since this is 2 times the integer $-s$, it is even, as desired.

Exercise

Use a direct proof to show that the product of two odd numbers is odd.

- An odd number is one of the form $2n + 1$, where n is an integer. We are given two odd numbers, say $2a+ 1$ and $2b + 1$. Their product is $(2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$. This last expression shows that the product is odd, since it is of the form $2n + 1$, with $n = 2ab + a + b$.

Proof by Contraposition

- Proofs of theorems of this type that are not direct proofs, that is, that do not start with the hypothesis and end with the conclusion, are called indirect proofs.
- An extremely useful type of indirect proof is known as proof by contraposition.
- Proofs by contraposition make use of the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive $\neg q \rightarrow \neg p$

Proof by Contraposition

- $p \to q$
  - $\neg q \to \neg p$
    - Take $\neg q$ as a hypothesis
    - Then show that $\neg p$ must follow

Example

Prove that if n is an integer and 3n + 2 is odd, then n is odd.

- We first attempt a direct proof. To construct a direct proof, we first assume that 3n + 2 is an odd integer. This means that 3n + 2 = 2k + 1 for some integer k. Can we use this fact to show that n is odd? We see that 3n + 1 = 2k, but there does not seem to be any direct way to conclude that n is odd. Because our attempt at a direct proof failed, we next try a proof by contraposition.

Example

- Assume that the conclusion of the conditional statement "If 3n + 2 is odd, then n is odd" is false; namely, assume that n is even.
- Then, by the definition of an even integer, n = 2k for some integer k.
- Substituting 2k for n , we find that

$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1).$$

- This tells us that 3n + 2 is even
- This is the negation of the hypothesis of the theorem.
- Negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true.

Proof by Contradiction

If we can show that $\neg p \rightarrow (r \wedge \neg r)$ is true for some proposition *r*, we can prove that *p* is true

Ex. Show that at least four of any 22 days must fall on the same day of the week.

Proof by Contradiction

Solution: Let p be the proposition "At least four of 22 chosen days fall on the same day of the week." Suppose that $\neg p$ is true. This means that at most three of the 22 days fall on the same day of the week. Because there are seven days of the week, this implies that at most 21 days could have been chosen because for each of the days of the week, at most three of the chosen days could fall on that day. This contradicts the hypothesis that we have 22 days under consideration. That is, if r is the statement that 22 days are chosen, then we have shown that $\neg p \rightarrow (r \wedge \neg r)$. Consequently, we know that p is true. We have proved that at least four of 22 chosen days fall on the same day of the week. ◀

Proof by contradiction

- Proof by contradiction can be used to prove conditional statements. In such proofs, we first assume the negation of the conclusion. We then use the premises of the theorem and the negation of the conclusion to arrive at a contradiction.