Discrete Mathematics

CMP-101

Lecture 8

Sets, Subsets, Power Sets, Set Operations and Identities, Computer Representation of Sets

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Outline

Sets

- Definition 1: A *set* is an unordered collection of objects
- Definition 2: Objects in a set are called *elements*, or *members* of the set.
  - $a \in A$, $a \notin A$
  - V = {a, e, i, o, u}
  - O = {1, 3, 5, 7, 9}

    or $O = \{x | x \text{ is an odd positive integer less than } 10\}$

    or $O = \{x \in \mathbf{Z}^{+} | x \text{ is odd and } x < 10\}$

Some Common Sets

- **N**={0, 1, 2, 3, …}, natural numbers
- **Z**={…,-2, -1, 0, 1, 2, …}, integers
- $\mathbf{Z}^+$={1, 2, 3, …}, positive integers
- **Q**=$\{p/q \mid p \in \mathbf{Z},\ q \in \mathbf{Z}, \text{ and } q \neq 0\}$, rational numbers
- $\mathbf{Q}^+ = \{x \in \mathrm{R} \mid x = p/q$, for positive integers p and $q\}$
- **R**, real numbers

Remark

Note that the concept of a datatype, or type, in computer science is built upon the concept of a set. In particular, a datatype or type is the name of a set, together with a set of operations that can be performed on objects from that set. For example, boolean is the name of the set {O, I } together with operators on one or more elements of this set, zuch as AND, OR, and NOT.

Equal Sets

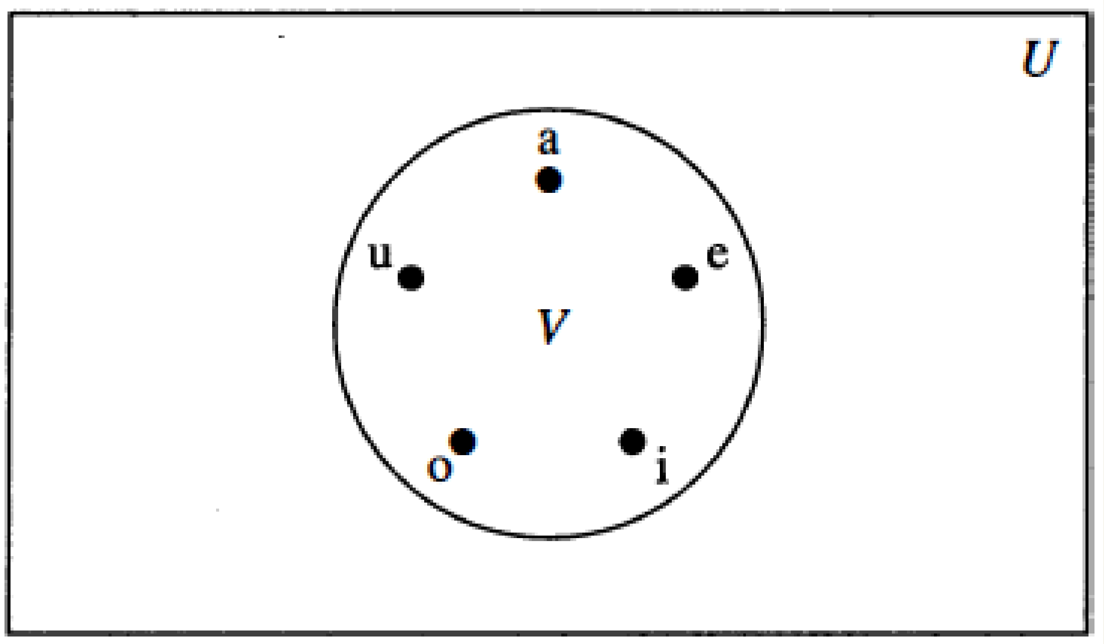
- Definition 3: Two sets are *equal* if and only if they have the same elements.
  $A = B$ iff $\forall x (x \in A \leftrightarrow x \in B)$
- $A = \{1,3,5,7\}$
- $B = \{7,3,1,5\}$
- A=B

Venn diagram

- Sets can be represented graphically using Venn diagrams, named after the English mathematician John Venn
- In Venn diagrams the universal set U, which contains all the objects under consideration, is represented by a rectangle.
- Circles or other geometrical figures are used to represent sets.

Example of Venn diagram

- Draw a Venn diagram that represents U, the set of vowels in the English alphabet.



Subset

- Definition 4: The set A is a subset of B if and only if every element of A is also an element of B.

  $A \subseteq B$ iff $\forall x(x \in A \rightarrow x \in B)$

- Theorem 1: For every set S,

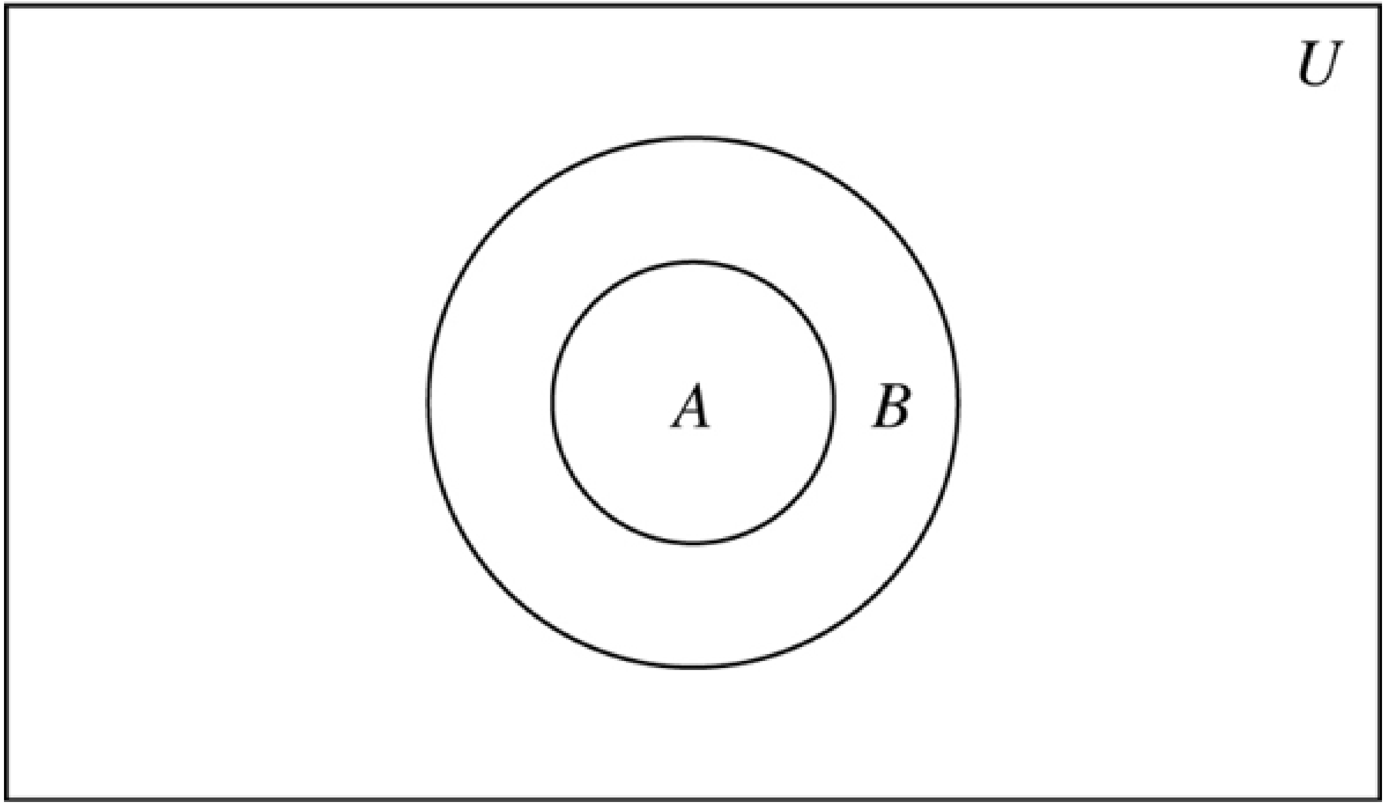
  (1) $\emptyset \subseteq S$

  (2) $S \subseteq S$.

Proper Subset

- When we wish to emphasize that a set A is a subset of the set B but that $A \neq B$, we write $A \subset B$ and say that A is a proper subset of B. For $A \subset B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A . That is, A is a proper subset of B if

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$



Venn Diagram Showing that *A* Is a Subset of *B*.

Equality of Sets

- If $A \subseteq B$ and $B \subseteq A$, then $A=B$
- Sets may have other sets as members
  - $A=\{\emptyset, \{a\}, \{b\}, \{a,b\}\}$

    $B=\{x \mid x \text{ is a subset of the set } \{a,b\}\}$
  - $A=B$

Cardinality of Set

If there are exactly n distinct members in the set S (n is a nonnegative integer), we say that S is a finite set and that n is the cardinality of S.

$|S| = n$

$|\varnothing| = 0$

For Example

Let A be the set of odd positive integers less than 10. Then $|A| = 5$.

Infinite Set

A set is said to be infinite if it is not finite.

For Example

The set of positive integers is infinite.

Z^+

Power Set

- Definition 7: The *power set* of S is the set of all subset of the set S. *P(S)*

  P({0,1,2})

- The power set P ({ O, 1 , 2 }) is the set of all subsets of {0, 1 , 2 } . Hence,
- P ({0, 1 , 2 }) = {∅ , {0}, { l } , {2 } , {0, 1} , {0,2} , { l , 2 } , {0, 1 , 2 } } .

Power Set

What is the power set of the empty set?

$P(\varnothing)=\{\varnothing\}$

What is the power set of the set $\{\varnothing\}$?

- $P(\{\varnothing\})=\{\varnothing,\{\varnothing\}\}$

- If a set has n elements, then its subset has 2^n elements.

Cartesian Products

- *Ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_i as its ith element for $i=1, 2, \ldots, n$.
- *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

  $A \times B = \{(a, b) | a \in A \wedge b \in B\}$
  - E.g. $A = \{1, 2\}$, $B = \{a, b, c\}$
  - $A \times B$ and $B \times A$ are not equal, unless A=∅ or B=∅ or A=B

Set Operations: *Union*

- Definition 1: The *union* of the sets *A* and *B*, denoted by $A \cup B$, is the set containing those elements that are either in *A* or in *B*, or in both.
  - $A \cup B = \{x | x \in A \vee x \in B\}$
- The union of the sets { l , 3 , 5 } and { l , 2, 3 } is the set { l , 2 , 3 , 5 } ; that is,

  { l , 3 , 5 } U { l , 2, 3 } ={ l , 2, 3 , 5 } .

Set Operations: *Intersection*

- Definition 2: The *intersection* of the sets *A* and *B*, denoted by $A \cap B$, is the set containing those elements in both *A* and *B*.
  - $A \cap B = \{x | x \in A \wedge x \in B\}$

The intersection of the sets { I , 3 , 5 } and { I , 2 , 3 } is the set { I , 3 } ; that is, { I , 3 , 5 } n { l , 2, 3 } ={ I , 3 } .

Set Operations: *Disjoint*

- Definition 3: Two sets are *disjoint* if their intersection is the empty set.

Let $A = \{ I , 3 , 5 , 7, 9\}$ and

$B = \{2, 4, 6, 8 , 10\}$

Because $A \cap B = \varnothing$, A and B are disjoint.

Set Operations: Principle of inclusion-exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Principle of inclusion-exclusion

Set Operations: *Difference*

- Definition 4: The *difference* of the sets A and B, denoted by A-B, is the set containing those elements that are in A but not in B.
  - Complement of B with respect to A
  - $A-B=\{x|x \in A \wedge x \notin B\}$

Set Operations: *Complement*

- Definition 5: Let U be the universal set. The complement of the set A, denoted by $\bar{A}$, is the complement of A with respect to U.
- $\bar{A} = \{x \mid x \notin A\}$

Set Identities

Identity	Name
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	Complement laws

Computer Representation of Sets

- The operations of computing the union, intersection, or difference of two sets would be time-consuming, because each of these operations would require a large amount of searching for elements
- We will present a method for storing elements using an arbitrary ordering of the elements of the universal set.
- This method of representing sets makes computing combinations of sets easy.

Computer Representation of Sets

- Assume that the universal set U is finite(Fits in Computer Memory)
- First, specify an arbitrary ordering of the elements of U, for instance a_1 , a_2 , . . . , a_n . Represent a subset A of U with the bit string of length n
- The ith bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A .
- See Example

Computer Representation of Sets

- Let U = { l , 2, 3 , 4, 5 , 6, 7 , 8 , 9, 10} , and the ordering of elements of U has the elements in increasing order; that is, a_i = i . What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U ?
- The bit string that represents the set of odd integers in U, namely, { I , 3 , 5 , 7, 9}, has a one bit in the first, third, fifth, seventh, and ninth positions, and a zero elsewhere. It is
  - 1 0 1 0 1 0 1 0 1 0.

Computer Representation of Sets

- Similarly, we represent the subset of all even integers in U, namely, {2, 4, 6, 8, 1 0} , by the string

  0 1 0 1 0 1 0 1 0 1

- The set of all integers in U that do not exceed 5 , namely, { l , 2 , 3 , 4 , 5 } , is represented by the string

  1 1 1 1 1 0 0 0 0 0.