Discrete Mathematics

CMP-101

Lecture 9

Functions, One-to-One and On-to Functions, Inverse Functions, Compositions of Functions

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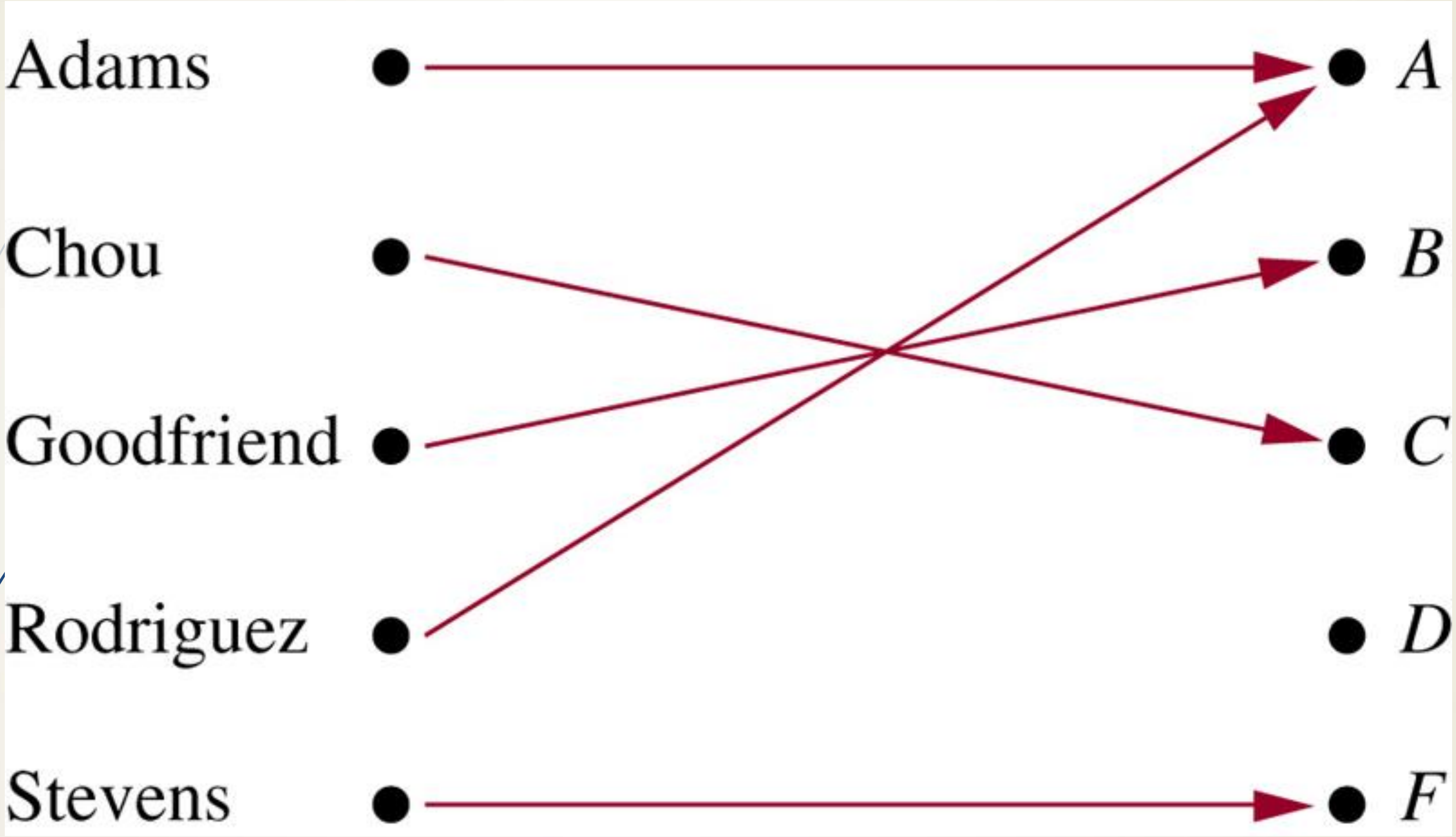
Functions

- In many instances we assign to each element of a set a particular element of a second set
- For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set { A , B , C , D , F } . And suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens.



Assignment of Grades in a Discrete Mathematics Class.

Functions

- In discrete mathematics functions are used in the definition of such discrete structures as sequences and strings.
- Functions are also used to represent how long it takes a computer to solve problems of a given size
- Many computer programs and subroutines are designed to calculate values of functions

Definition 1:

- A *function f* from *A* to *B* is an assignment of exactly one element of *B* to each element of *A*.
- $f: A \to B$
- Remark: Functions are sometimes also called mappings or transformations.

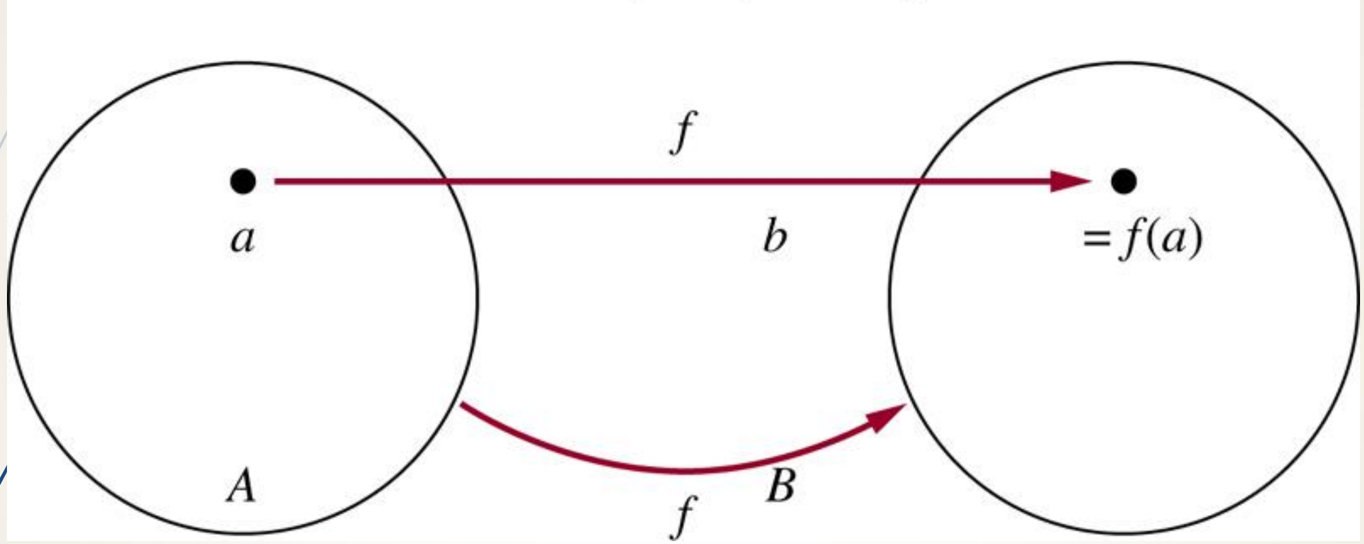


FIGURE 2 The Function f Maps *A* to *B*.

A Word:

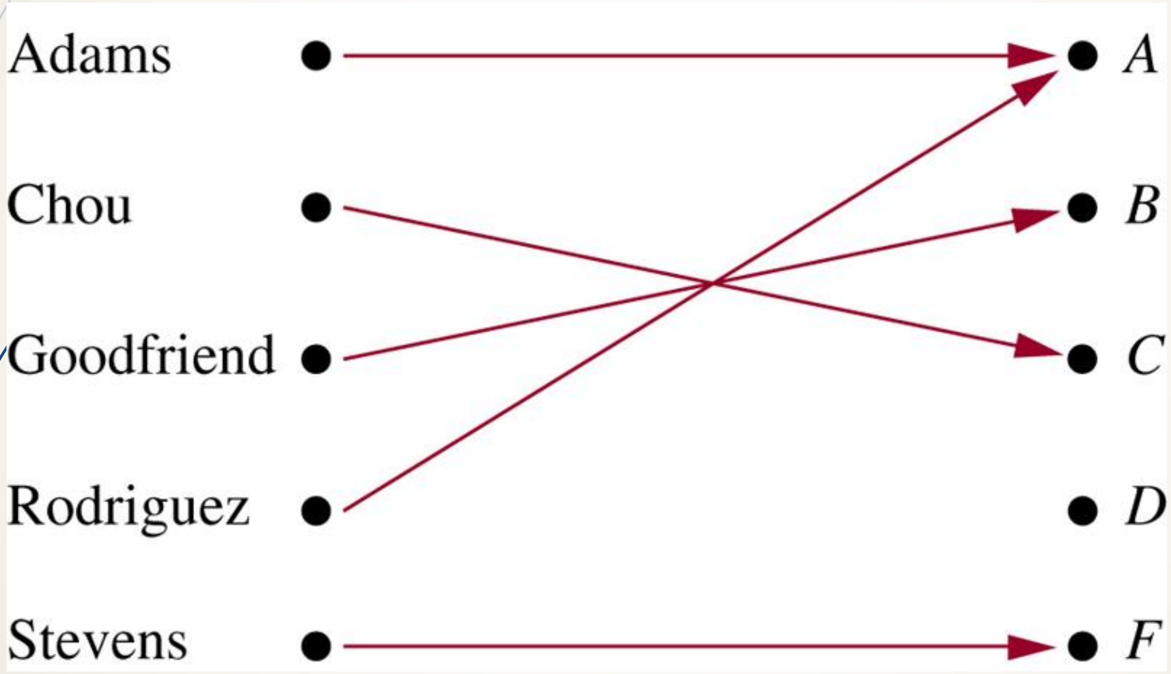
- A function $f : A \rightarrow B$ can also be defined in terms of a relation from A to B.
- A relation from A to B is just a subset of A x B .
- A relation from A to B that contains one, and only one, ordered pair (a , b) for every element a $\in$ A, defines a function f from A to B .
- This function is defined by the assignment f(a) = b, where (a,b) is the unique ordered pair in the relation that has a as its first element.

Some Important Definitions:

- If *f* is a function from A to B
  - *A*: *domain* of *f*
  - *B*: *codomain* of *f*
  - *f(a)=b*, *a*: *preimage* of *b*, *b*: *image* of *a*
  - *Range* of *f*: the set of all images of elements of *A*
  - If *f* is a function from A to B , we say that f maps A to B .

Example

What are the domain, codomain, and range of the function that assigns grades to students.



Assignment of Grades in a Discrete Mathematics Class.

Solution

- Let G be the function that assigns a grade to a student
- **The domain** of G is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}
- **Codomain** is the set {A, B , C , D , F }
- **The range** of G is the set {A, B , C , F}

Example

- Let f: Z $\rightarrow$ Z assign the square of an integer to this integer.
- $f(x) = x^2$, where the domain of f is the set of all integers, we take the the codomain of f to be the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, {0,1,4,9, . . . }

Sum and Product of Functions

- Let f_1 and f_2 be functions from A to **R**. f_1+f_2 and f_1f_2 are also functions from A to **R**:
  - $(f_1+f_2)(x) = f_1(x)+f_2(x)$
  - $(f_1f_2)(x)=f_1(x)f_2(x)$

Example

Let f_1 and f_2 be functions from **R** to **R** such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution: From the definition of the sum and product of functions, it follows that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

and

$$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4.$$

Definition 4:

- $f: A \to B$, S is a subset of A. The image of S under the function f is:
- $f(S)=\{f(s)|s \in S\}$

Let A = {a, b, c, d, e} and B = {l, 2, 3, 4}
f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1, f(e) = 1 .
The image of the subset S = {b,c,d } is the set
f(S) = { 1 , 4}

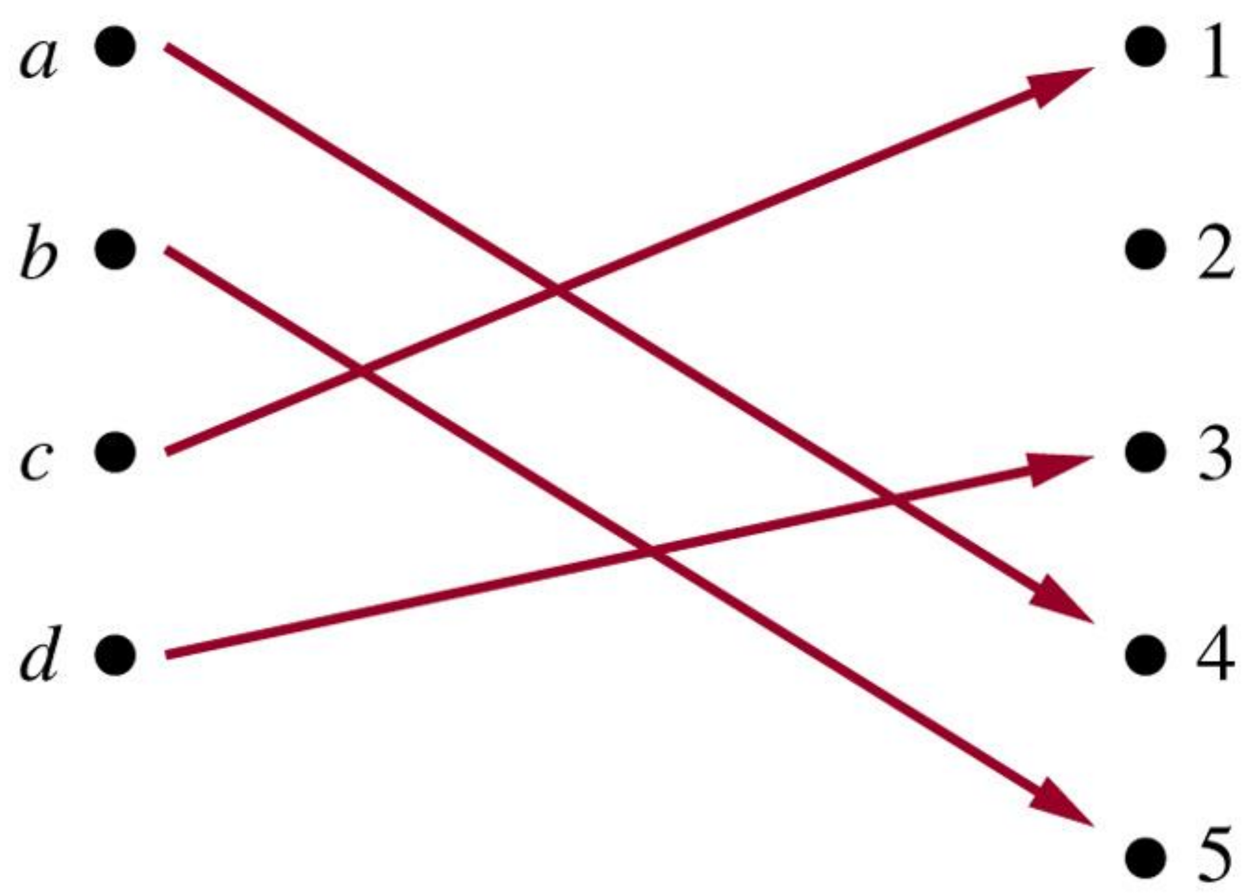
One-to-One Function

- A function f is *one-to-one* or *injective*, iff $f(a)=f(b)$ implies that $a=b$ for all a and b in the domain of f.
  - $\forall a \, \forall b(f(a)=f(b) \rightarrow a=b)$
  - $\forall a \, \forall b(a \neq b \rightarrow f(a) \neq f(b))$

Example of One-to-One Function

- Determine whether the function *f* from {a, b, c, d} to { 1 , 2, 3 , 4, 5 } with f(a) = 4, f(b) = 5, f(c) = 1, and f(d) = 3 is one-to-one.

- The function f is one-to-one because f takes on different values at the four elements of its domain.



A One-to-One Function.

Example of One-to-One Function

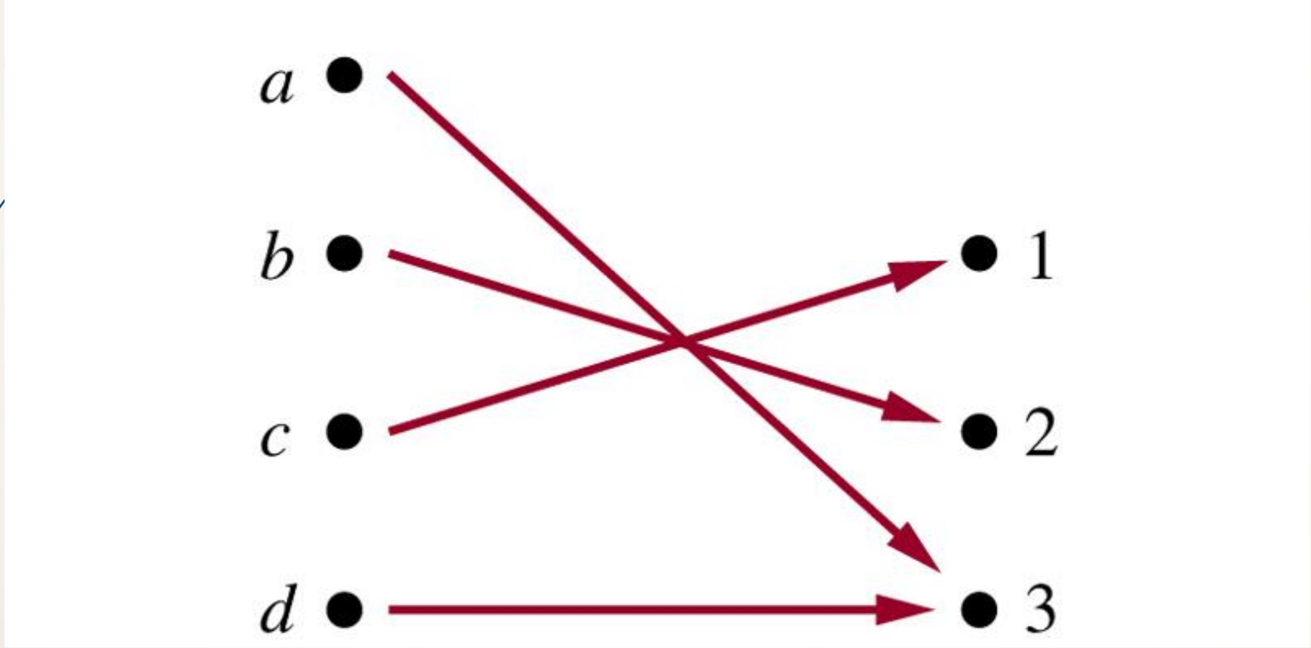
- Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.
- The function $f(x) = x^2$ is not one-to-one because, for instance,
- $f(1) = f(-1) = 1$, but $1 \neq -1$

On-to Function

- A function f is *onto* or *surjective*, iff for every element $b \in B$ there is an element $a \in A$ with $f(a)=b$.
  - $\forall y \exists x(f(x)=y)$ or
  - $\forall a \forall b(a \neq b \rightarrow f(a) \neq f(b))$

Example

- Let f be the function from {a , b, c, d} to { 1 , 2, 3} defined by *f*(a) = 3, *f*(b) = 2, *f*(c) = 1 , and *f*(d) = 3 . Is *f* an onto function?

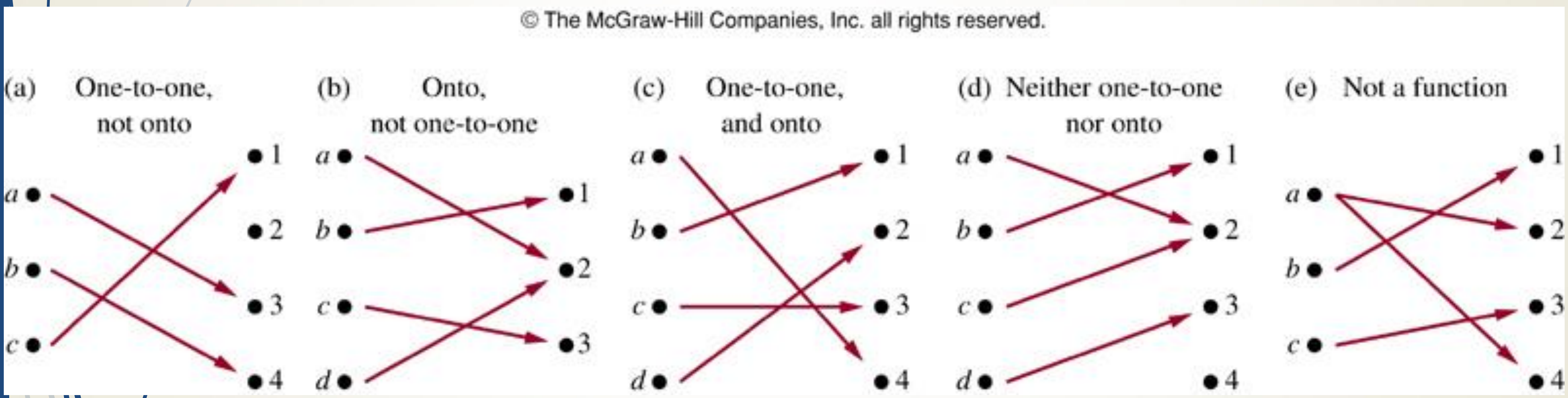


Example

- Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

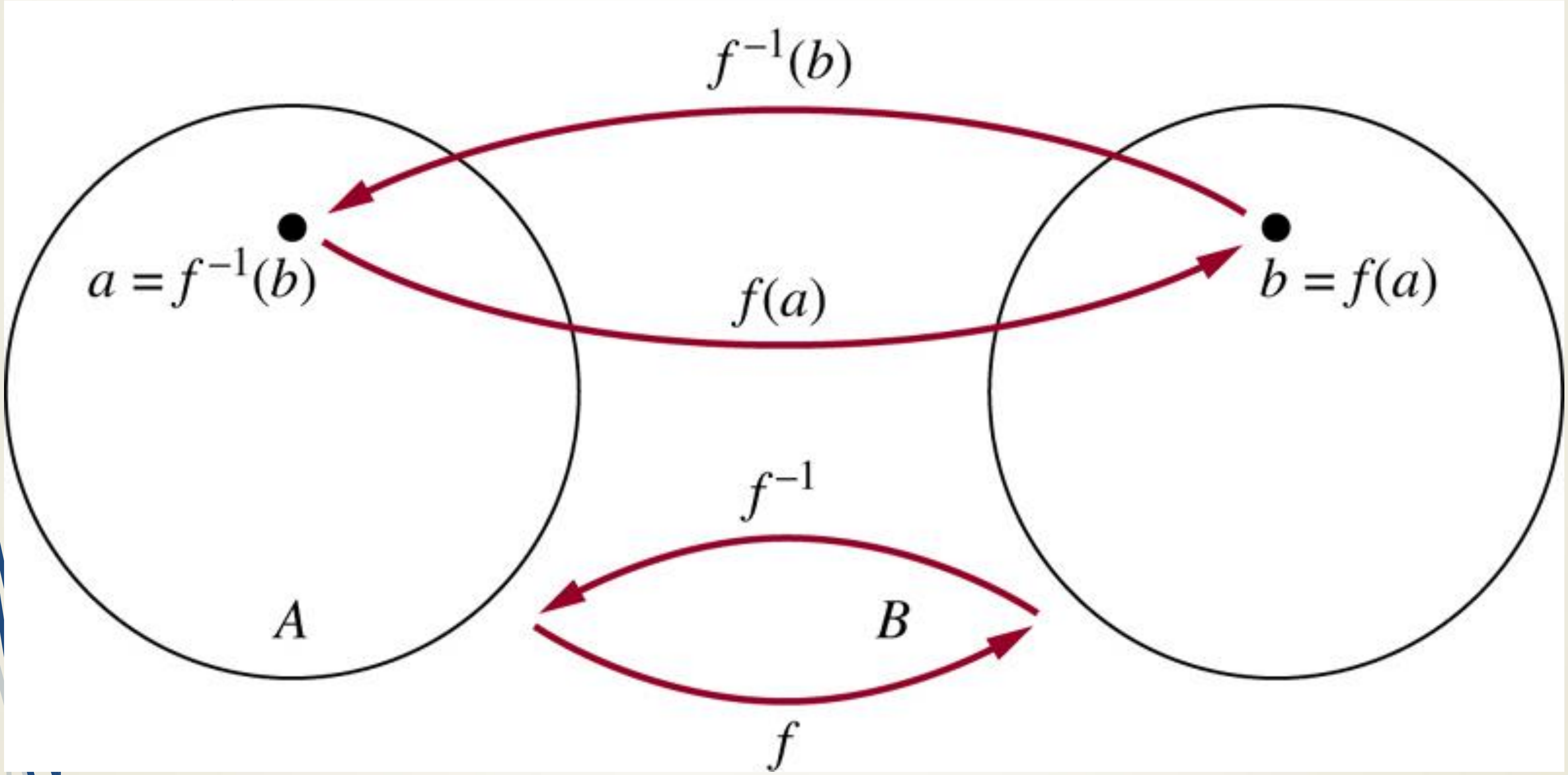
- The function f is not onto because there is no integer x with $x^2 = -1$, for instance

Examples of Different Types of Correspondences.



Inverse Functions

- Let f be a one-to-one correspondence from A to B. The inverse function of f is the function that assigns to an element b in B the unique element a in A such that f(a)=b.
  - $f^{-1}(b)=a$ when $f(a)=b$



$f^{-1}(b)$
$a = f^{-1}(b)$
$f(a)$
$b = f(a)$
f^{-1}
A
B
f

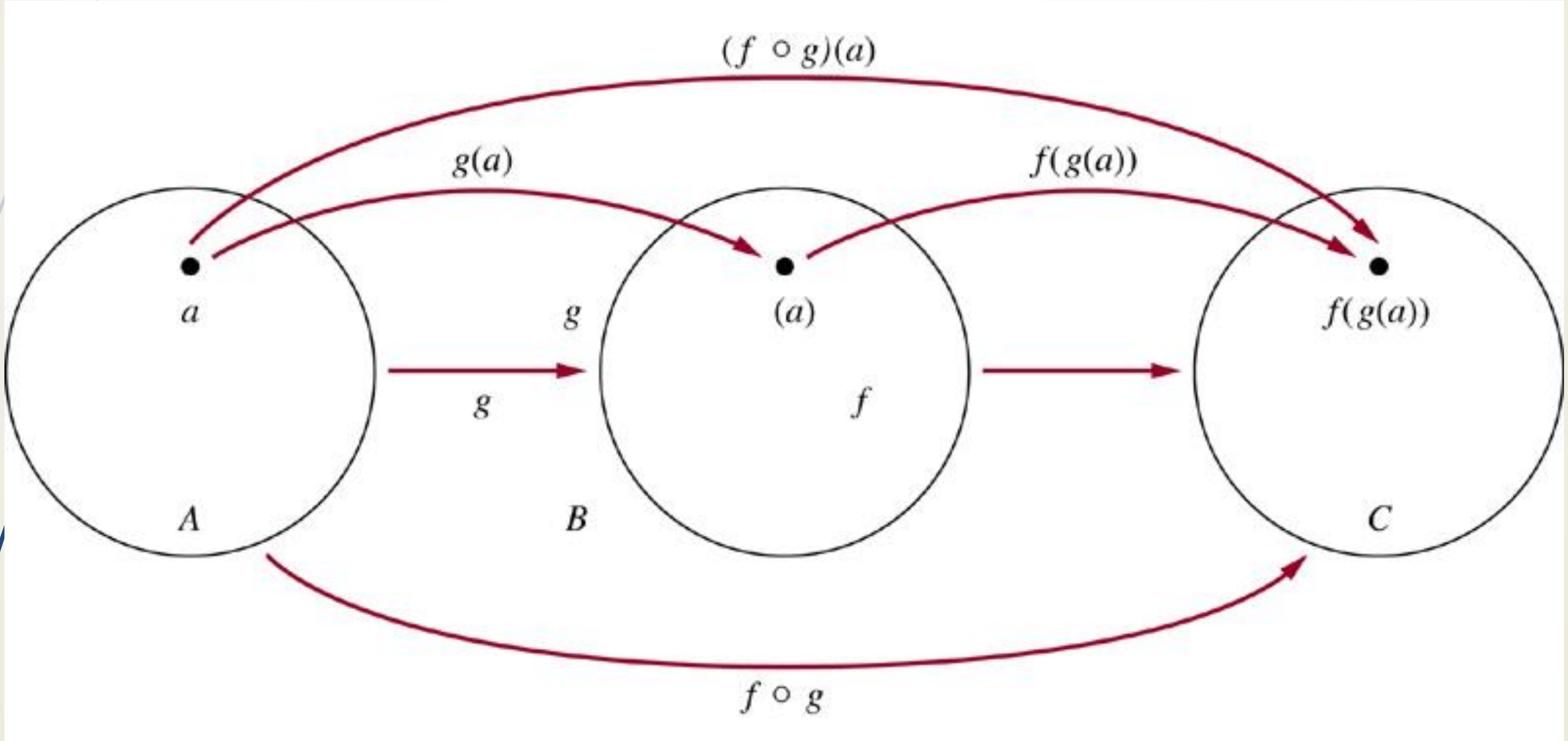
Invertible

- A one-to-one correspondence is called **invertible** because we can define an inverse of this function. A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

The composition of functions

- Let g be a function from A to B, and f be a function from B to C. The composition of functions f and g, denoted by $f \circ g$, is defined by:
  - $f \circ g\,(a) = f(g(a))$
  - $f \circ g$ and $g \circ f$ are not equal

The Composition of the Functions *f* and *g*



A Word:

- Note that the composition f o g cannot be defined unless the range of g is a subset of the domain of f

Example

- Let g be the function from the set {a , b, c} to itself such that g(a) = b, g(b) = c, and g(c) = a. Let f be the function from the set {a , b , c} to the set { l , 2 , 3 } such that f(a) = 3 , f(b) = 2, and f(c) = 1. What is the composition of f and g, and what is the composition of g and f?
- The composition f o g is defined by (f o g)(a) = f(g(a)) = f(b)= 2, (f o g) (b) = f(g(b)) = f(c) = 1 , and (*fog*)(c)=f(g(c))=f(a) = 3 .
- Note that g o f is not defined, because the range of f is not a subset of the domain of g .